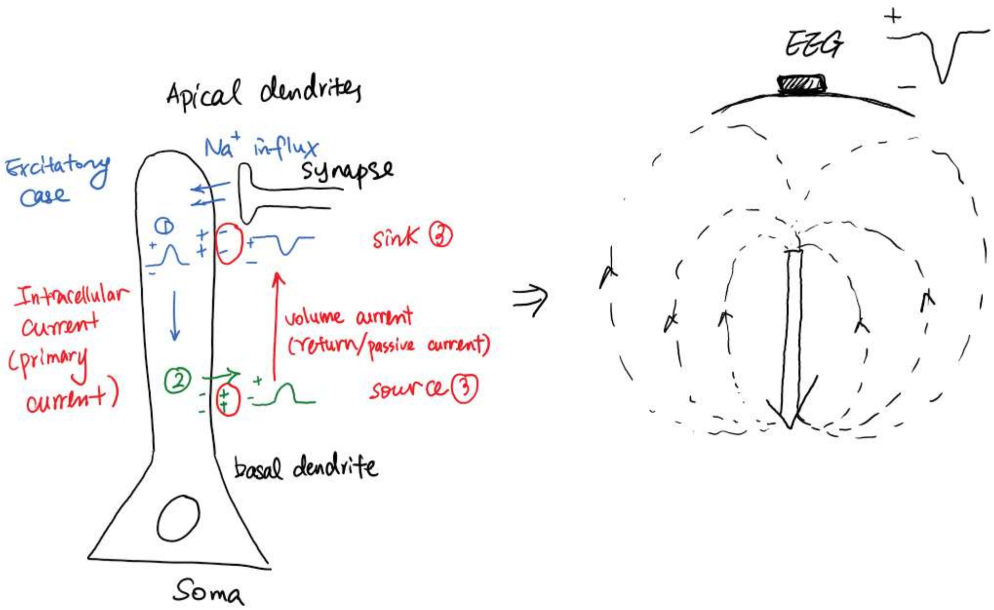
Equivalent Current Dipole

Thursday, August 15, 2019 1:18 PM

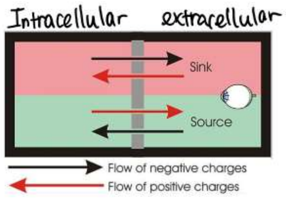
Anatomy



① Accumulate positive charges

② Equivalent capacitive efflux of positive charges

③ Intracellular extracellular



https://en.wikipedia.org/wiki/Current_sources_and_sinks

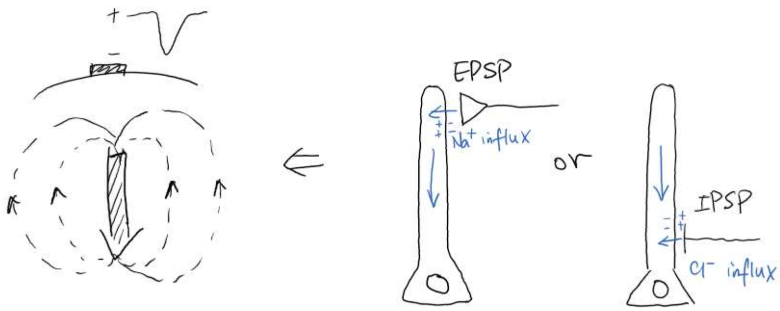
At the synapse, because of Na+ influx, there is an accumulation of positive charges inside the membrane and consequently negative charges accumulate in the extracellular space. Because of the electric field the charges generated, there is a intracellular current flow towards the soma and this is the **primary current** that gives rise the return current and further the EEG. The return current is the extracellular volume current with opposite direction from the intracellular current that completes the ionic flow loop. Because of these currents, there will be a efflux of positive changes at the basal dendrite as part of the loop to balance the charges. According to the definition of current source and sink, we have a current source at the apical dendrite and current sink at the basal dendrite.

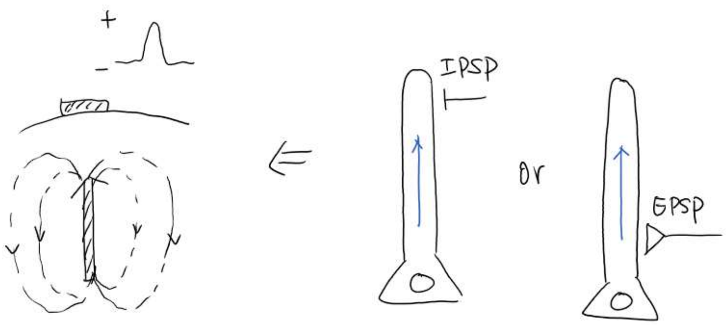
When thousands of dendrites are aligned and synchronized, it can generate EEG measurable signals. We can model the activity as a current dipole pointing downwards. This will give a negative deflection of the scalp potential imposed by the return current. And what we want to estimate is the source of this return current, which is the primary current.

https://www.fil.ion.ucl.ac.uk/spm/course/slides14-meeg/01_MEEG_What_are_we_measuring_with_MEEG.pdf

- 50 000 cells succient to generate a dipolar source of 10nAm

- About 10 000 synchronous neurons could yield an MEG measurable signal





Mathematical Modeling

=0 for quasi-static approximation

Maxwell equations:

$$\nabla \times \mathbf{E} = \boxed{-\frac{\partial \mu_o \mathbf{H}}{\partial t}}$$

Faraday's law: changing magnetic field generates electric field

$$\nabla \times \mathbf{H} = \mathbf{J} + \boxed{\frac{\partial \epsilon_o \mathbf{E}}{\partial t}}$$

Ampere-Maxwell: changing electric field generates magnetic field

$$\nabla \cdot \epsilon_o \mathbf{E} = \rho$$

Charges generates electric field

$$\nabla \cdot \mu_o \mathbf{H} = 0$$

No monopole

Also we have $\vec{E} = -\nabla\phi$, $\vec{J} = \sigma\vec{E}$ (Ohm's law)

only for (quasi-static)

from a theorem from vector calculus,

$$\nabla \times \vec{E} = 0 \iff \exists \phi, \ E = -\nabla\phi$$

if & only if

Continuity equation:

- **Divergence:** The physical significance of the **divergence** of a vector field is the rate at which "density" exits a given region of space. The definition of the divergence therefore follows naturally by noting that, in the absence of the creation or destruction of matter, **the density within a region of space can change only by having it flow into or out of the region**. By measuring the **net flux** of content passing through a surface surrounding the region of space, it is therefore immediately possible to say **how the density of the interior has changed**. This property is fundamental in physics, where it goes by the name "principle of continuity." When stated as a formal theorem, it is called the divergence theorem, also known as Gauss's theorem. In fact, the definition in equation is in effect a statement of the divergence theorem.

$$\nabla\cdot\mathbf{F}\equiv\lim_{V\to 0}\frac{\oint_S \mathbf{F}\cdot d\,\mathbf{a}}{V}$$

From <http://mathworld.wolfram.com/Divergence.html>

- Two different ways of deriving it:
- https://en.wikipedia.org/wiki/Current_density

Continuity equation [edit]

Main article: Continuity equation

Since charge is conserved, current density must satisfy a continuity equation. Here is a derivation from first principles.[9]

The net flow out of some volume *V* (which can have an arbitrary shape but fixed for the calculation) must equal the net change in charge held inside the volume:

$$\int_S \mathbf{j}\cdot \mathrm{d}\mathbf{A} = -\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\,\mathrm{d}V = -\int_V \frac{\partial\rho}{\partial t}\,\mathrm{d}V$$ definition

where *ρ* is the charge density, and *dA* is a surface element of the surface *S* enclosing the volume *V*. The surface integral on the left expresses the current *outflow* from the volume, and the negatively signed volume integral on the right expresses the *decrease* in the total charge inside the volume. From the divergence theorem:

$$\int_S \mathbf{j}\cdot \mathrm{d}\mathbf{A} = \int_V \nabla\cdot\mathbf{j}\,\mathrm{d}V$$ divergence theorem

Hence:

$$\int_V \nabla\cdot\mathbf{j}\,\mathrm{d}V = -\int_V \frac{\partial\rho}{\partial t}\,\mathrm{d}V$$

This relation is valid for any volume, independent of size or location, which implies that:

$$\nabla\cdot\mathbf{j} = -\frac{\partial\rho}{\partial t}$$

and this relation is called the continuity equation.[13][14]

- From Maxwell equation:

From $\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial \varepsilon_0\vec{E}}{\partial t}$

$$\underbrace{\vec{\nabla}\cdot(\vec{\nabla}\times\vec{H})}_{0} = \vec{\nabla}\cdot\vec{J} + \underbrace{\vec{\nabla}\cdot\frac{\partial \varepsilon_0\vec{E}}{\partial t}}_{\frac{\partial}{\partial t}\underbrace{\vec{\nabla}\cdot\varepsilon_0\vec{E}}_{\rho}}$$

$$\Downarrow$$

$$\vec{\nabla}\cdot\vec{J} + \frac{\partial\rho}{\partial t} = 0$$

- In my own words, the divergence of the electric field represents how the electric charges change in a volume in total. And the divergence of the current density represents how the charge density change over a unit time. (Think about the definition of the current density)
- The continuity equation is true universally.
- My thinking is current 0? -- current over the whole brain surface is 0, but we cannot say at each given location the current (density) is 0.

Poisson equations:

$$\vec{J}_{tot} = \vec{J}^p + \vec{J}^r \xrightarrow{\vec{\nabla}\cdot\vec{J}_{tot} = 0} \vec{\nabla}\cdot\vec{J}^p = -\vec{\nabla}\cdot\vec{J}^r = -\sigma(\vec{\nabla}\cdot\vec{J}^r) = \sigma\nabla^2\phi$$

primary current ($\vec{J}^p$), return current ($\vec{J}^r$)

Ohm's law, quasi-static Maxwell

↓ solve Possion equation:

$$4\pi\sigma\phi = -\int_V \left(\frac{1}{r}\right)\boxed{\vec{\nabla}\cdot\vec{j}^p}\,dv \qquad -\nabla\cdot\vec{j}^p = I_F$$

$$\downarrow \quad \vec{\nabla}\cdot\left(\frac{\vec{j}}{r}\right) = \nabla\left(\frac{1}{r}\right)\cdot\vec{j} + \vec{\nabla}\cdot\vec{j}\cdot\frac{1}{r}$$

$$\boxed{4\pi\sigma\phi = \int\nabla\left(\frac{1}{r}\right)\cdot\vec{j}^p\,dV} - \underbrace{\int\vec{\nabla}\cdot\left(\frac{\vec{j}^p}{r}\right)dv}_{=0} \Rightarrow \iiint\vec{\nabla}\left(\frac{\vec{j}^p}{r}\right)dv = \underbrace{\oint\frac{\vec{j}^p}{r}\cdot d\vec{s}}$$

ϕ in infinite homogeneous

no source of the surface $=0$

boundary condition

$$\phi(s_i^+) = \phi(s_i^-), \quad \sigma_i^+\nabla\phi(s_i^+)\cdot d\vec{n}_i = \sigma_i^-\nabla\phi(s_i^-)\cdot d\vec{n}_i$$

Green's theorem

$$4\pi\sigma(r)\phi(r) = \int_V \vec{j}^p(r')\vec{\nabla}'\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)dv - \sum_i(\sigma_i^- - \sigma_i^+)\int_{S_i}\phi(s_i)\vec{\nabla}'\left(\frac{1}{|\vec{r}-\vec{s}'|}\right)d\vec{S}_i$$

intergal over brain volumn

derivative over r'

derivate over S'

↓ look at scalp surface

$$V(\vec{s}) = V_\infty(\vec{s}) - \frac{1}{2\pi}\sum_{l}^{N_S}\frac{\sigma_l^- - \sigma_l^+}{\sigma_k^- + \sigma_k^+}\int_{S_l'} V(\vec{s}')\vec{\nabla}'\left(\frac{1}{|\vec{s}-\vec{s}'|}\right)\vec{n}(\vec{s}')\,dS_l' \qquad \textbf{28.15}$$

where $V_\infty(\vec{s})$ is the potential due to $\vec{j}_f$ in a conductor of infinite extent and homogeneous conductivity $(\sigma_k^- + \sigma_k^+)/2$:

$$V_\infty(\vec{s}) = \frac{1}{2\pi(\sigma_k^- + \sigma_k^+)}\int_v \vec{j}_f(\vec{r}')\vec{\nabla}'\left(\frac{1}{|\vec{s}-\vec{r}'|}\right)dv \qquad \textbf{28.16}$$

Solve 28.15 for BEM model:

$$V(\vec{s}) = V_\infty(\vec{s}) - \frac{1}{2\pi}\sum_{l=1}^{N_S}\frac{\sigma_l^- - \sigma_l^+}{\sigma_k^- + \sigma_k^+}$$

$$\sum_{m=1}^{N_{tr}^{(l)}}\int_{\Delta_m^{(l)}} V(\vec{s}')\vec{\nabla}'\left(\frac{1}{|\vec{s}-\vec{s}'|}\right)\vec{n}(\vec{s}')\,dS' \qquad \textbf{28.19}$$

Representation over the triangle.

Over the triangle

We can use centre of gravity (evaluate V at the centre of gravity of each triangle and consider this value constant

We can use centre of gravity (evaluate V at the centre of gravity of each triangle and consider this value constant over the triangle), constant potential at vertices (the potential over the triangle is supposed to be constant and equal to the mean of the potential at its vertices), linear potential at vertices (potential varies linearly over the triangle) to find V(s).
Take CoG for instance,

$$\begin{aligned}V_\infty(\vec{s}) &= \frac{1}{2\pi(\sigma_k^- + \sigma_k^+)} \int_v \frac{\vec{s}-\vec{r}'}{|\vec{s}-\vec{r}'|^3} \sum_{i=1}^{N_j} \vec{j}_f(\vec{r}_i)\delta(\vec{r}'-\vec{r}_i)\,dv \\ &= \frac{1}{2\pi(\sigma_k^- + \sigma_k^+)} \sum_{i=1}^{N_j} \vec{j}_f(\vec{r}_i) \int_v \frac{\vec{s}-\vec{r}'}{|\vec{s}-\vec{r}'|^3}\delta(\vec{r}'-\vec{r}_i)\,dv \\ &= \frac{1}{2\pi(\sigma_k^- + \sigma_k^+)} \sum_{i=1}^{N_j} \frac{\vec{s}-\vec{r}_i}{|\vec{s}-\vec{r}_i|^3}\vec{j}_f(\vec{r}_i) \qquad \text{28.21}\end{aligned}$$

$$\int_{\Delta_m^{(l)}} V(\vec{s}')\vec{\nabla}'\left(\frac{1}{|\vec{s}-\vec{s}'|}\right)\vec{n}(\vec{s}')\,dS' = -V(\vec{s}'_{cog})\Omega^{(l,m)}(\vec{s}) \qquad \text{28.22}$$

where $\Omega^{(l,m)}(\vec{s})$ is the solid angle at $\vec{s}$ subtended by the triangle $\Delta_m^{(l)}$:

$$\Omega^{(l,m)}(\vec{s}) = -\int_{\Delta_m^{(l)}} \vec{\nabla}'\left(\frac{1}{|\vec{s}-\vec{s}'|}\right)\vec{n}(\vec{s}')\,dS' = \int_{\Delta_m^{(l)}} \frac{\vec{s}'-\vec{s}}{|\vec{s}'-\vec{s}|^3}\vec{n}(\vec{s}')\,dS' \qquad \text{28.23}$$

← Solve using geometry information.

$$V(\vec{s}_{cog,p}) = V_\infty(\vec{s}_{cog,p}) + \frac{1}{2\pi}\sum_{l=1}^{N_S}\frac{\sigma_l^- - \sigma_l^+}{\sigma_k^- + \sigma_k^+}\sum_{m=1}^{N_{tr}^{(l)}} V(\vec{s}_{cog,m})\Omega^{(l,m)}(\vec{s}_{cog,p}) \qquad \text{28.24}$$

Matrix form solution:

the BEM problem can be expressed in matrix form:

$$\begin{bmatrix}\mathbf{v}_1\\ \mathbf{v}_2\\ \mathbf{v}_3\end{bmatrix} = \begin{bmatrix}\mathbf{B}_{11} & \mathbf{B}_{12} & \mathbf{B}_{13}\\ \mathbf{B}_{21} & \mathbf{B}_{22} & \mathbf{B}_{23}\\ \mathbf{B}_{31} & \mathbf{B}_{32} & \mathbf{B}_{33}\end{bmatrix}\begin{bmatrix}\mathbf{v}_1\\ \mathbf{v}_2\\ \mathbf{v}_3\end{bmatrix} + \begin{bmatrix}\mathbf{G}_1\\ \mathbf{G}_2\\ \mathbf{G}_3\end{bmatrix}[\mathbf{j}] \Leftrightarrow \mathbf{v} = \mathbf{B}\,\mathbf{v} + \mathbf{G}\,\mathbf{j} \qquad \text{28.35}$$

$\Rightarrow (I_{N_v} - B)\,v = Gj$

rank $N_v - 1$

↓ find p^t, so that $p^t v = 0$

$(I_{N_v} - B - 1_{N_v} p^t)\,v = Gj$

rank N_v

constrain, mean = 0

$p^t = [0\ 0 \cdots 0, \frac{1}{N_{v_3}}, \frac{1}{N_{v_3}} \cdots \frac{1}{N_{v_3}}]$

vertices in brain/skull

vertices on scalp

$v = (I_{N_v} - B - 1_{N_v} p^t)^{-1} G j$

scalp potential

current dipole

where:

- $\mathbf{v}_k$, an $N_{v_k} \times 1$ vector, contains the potential at the N_{v_k} nodal points of surface S_k: centre of gravity of each triangle for the CoG approximation or vertices of the triangles for the LPV approximation. $\mathbf{v}$ is $N_v \times 1$ vector with $N_v = N_{v_1} + N_{v_2} + N_{v_3}$.
- $\mathbf{B}_{kl}$, an $N_{v_k} \times N_{v_l}$ matrix, represents the influence of the potential of surface S_l on the potential of surface S_k. Its elements depend on the conductivity inside and outside the surfaces S_k and S_l, and on the solid angles used in the BEM Eqn. 28.24. $\mathbf{B}$ is an $N_v \times N_v$ matrix.
- $\mathbf{j} = [\vec{j}_1^t\ \vec{j}_2^t \ldots \vec{j}_{N_j}^t]^t$, a $3N_j \times 1$ vector, is the source distribution vector, where each $\vec{j}_n = [j_{n,x}\ j_{n,y}\ j_{n,z}]^t$ is an orientation-free source vector.
- $\mathbf{G}_k$, an $N_{v_k} \times 3N_j$ matrix, is the free space potential matrix depending on the location $\vec{r}_n$ of the sources $\vec{j}_n$, the nodal points on surface S_k and the conductivity inside and outside surface S_k (σ_k^- and σ_k^+). $\mathbf{G}$ is an $N_v \times 3N_j$ matrix.

Note: Solve Poisson equations:

$\nabla^2 u = v$

Divide v into δ functions, then if we can find

$\nabla^2 G(r-r') = \delta(r-r')$, we can solve possion's equation by

$$u(r) = \int G(r,r') v(r') d^3 r$$

It is known that $G(r,r') = -\frac{1}{4\pi}\frac{1}{|r-r'|}$, so

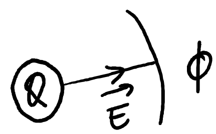
$$u(r) = -\int \frac{v(r')}{|r-r'|} d^3 r'$$

Dipole equations:

Monopole

duality

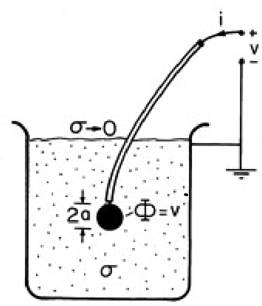
Charge



$$\vec{J} = \frac{I_0}{4\pi r^2} \cdot \vec{a}_r \qquad \vec{J} = \sigma \vec{E} \qquad \vec{E} = \frac{Q}{4\pi \varepsilon r^2} \cdot \vec{a}_r$$

$$\phi = \frac{I_0}{4\pi \sigma r} \qquad \phi = \frac{Q}{4\pi \varepsilon r}$$

Example:

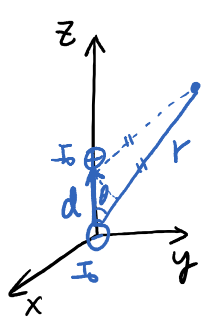


Dipole

No single isolated monopole current source because of the need to conserve charge.

Derivation idea 1:
http://web.mit.edu/6.013_book/www/chapter4/4.4.html.

$\vec{p} = I_0 \vec{d}$ dipole moments



$r_+ \approx r - d \cdot \cos\theta$

$r_- = r$

$\phi = \phi_{r_+} + \phi_{r_-}$

$= \frac{I_0}{4\pi\sigma}\left(\frac{1}{r_+} - \frac{1}{r_-}\right)$ $\quad \nabla\left(\frac{1}{r}\right)\cdot\vec{d}$

from $(a+b)^n = a^n + n a^{n-1} b + \cdots$

ignore b^m, $m > 2$

ignore b^m, $m > 2$

$$(r - d\cos\theta)^{-1} \approx \frac{1}{r} + \frac{1}{r^2} d\cos\theta$$

$$\phi = \frac{I_0}{4\pi\sigma}\left(\frac{1}{r} + \frac{1}{r^2} d\cos\theta - \frac{1}{r}\right) = \frac{I_0 d}{4\pi\sigma r^2}\cos\theta = \frac{P}{4\pi\sigma r^2} \cdot \underline{\underline{\cos\theta}}$$

Generalize to arbitrary dipole

$$\boxed{\phi = \frac{P}{4\pi\sigma r^2} \cdot \underline{\underline{\vec{a_d} \cdot \vec{a_r}}}}$$

Derivation idea 2:

If the two monopoles are overlapped, there will be no field at location r. --> $\phi(r)$ is the function of d. --> Use Taylor expansion: $\phi(r) = \phi(0) + \nabla\phi(r)\, dr$

$$\Phi_d = \boxed{\frac{\partial\left(\frac{I_o}{4\pi\sigma r}\right)}{\partial d}} d$$

→ why this term:

Because $d \ll r$, we can still use the monopole equation, However, the r here is a function of d, the equation actually is $\frac{I_0}{4\pi\sigma r(d)}$ (see deriv 1, use d to calculate r_+, r_-)

$$\Phi_d = \frac{p}{4\pi\sigma}\nabla\left(\frac{1}{r}\right) \cdot \bar{a}_d$$

- isotropic: having a physical property which has the same value when measured in different directions
- homogeneous--uniform